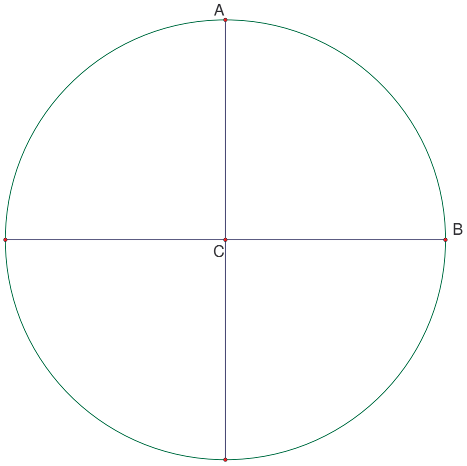
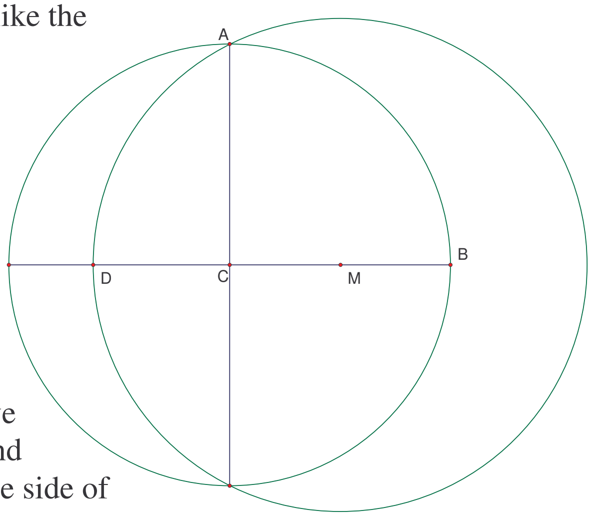
# How to Inscribe a Pentagon in a Circle with Geometer's Sketchpad (with help from the Golden Ratio)

In order to inscribe a pentagon in a circle, we have to come up with a right triangle where one of the legs has length 1 and the other one $\frac{1}{\varphi} = \varphi - 1$. To see why this is true, see the following pages.



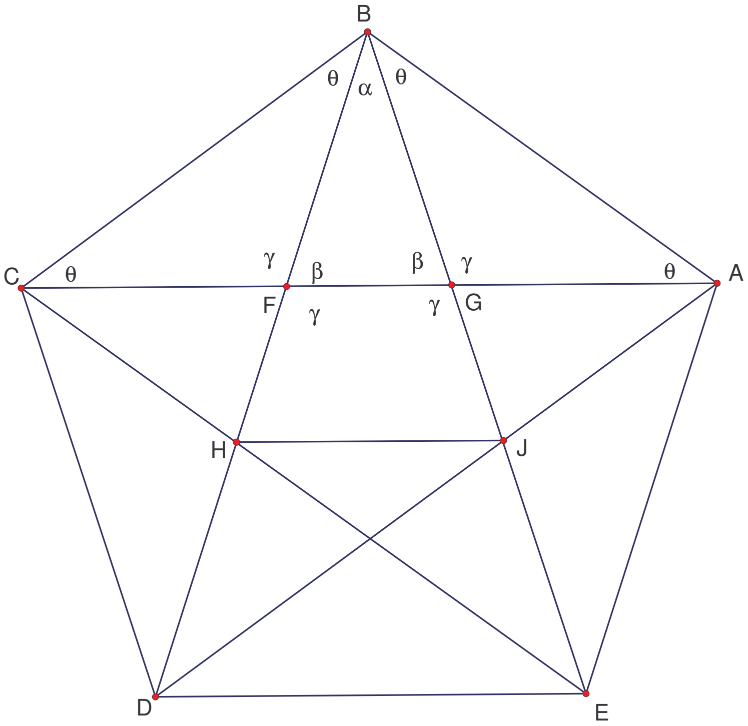
1. Draw a circle.
2. Draw two perpendicular diameters of the circle, as in the sketch at right. Think of the circle as a unit circle, so that $AC = BC = 1$.
3. Construct the midpoint $M$ of $CB$.
4. Draw a circle with center $M$ that goes through point $A$. Note that the radius of this circle is $\sqrt{\left(\frac{1}{2}\right)^2 + 1^2} = \frac{\sqrt{5}}{2}$.
5. Label the intersection of circle $M$ with line $\overleftrightarrow{BC}$ point $D$.
6. Now we have a sketch that looks something like the sketch at right. Note that $MD = MA = \frac{\sqrt{5}}{2}$, and that $MB = \frac{1}{2}$. So $DB = MD + MB = \frac{\sqrt{5}+1}{2} = \varphi$. Since $CB =$ 1 and $DC = DB - CB$, that means that $DC = \varphi - 1$.
7. Draw the segment from $A$ to $D$. We now have right triangle $\Delta ACD$, with legs of length 1 and $\varphi - 1$. This means that $AD$ is the length of the side of an inscribed pentagon.
8. Draw a circle with center $A$ that goes through $D$. Mark the intersection with circle $C$. Draw another circle with center at the intersection you just marked that goes through $A$. Again, mark the intersection with circle $C$. Keep on drawing circles and marking intersection points until you have the five vertices of your pentagon. Connect the vertices, and hide all the other stuff.



The theory behind the pentagon:

Let's look at some features of a regular pentagon, and how the number $\varphi$ (the golden ratio, or golden mean) relates to it. Recall that $\varphi$ is the positive solution to $\varphi^2 = \varphi + 1$, which is equivalent to $\varphi = 1 + \frac{1}{\varphi}$. This turns out to be $\varphi = \frac{1+\sqrt{5}}{2}$

Here's a look at a regular pentagon, along with its diagonals, forming a nice star:

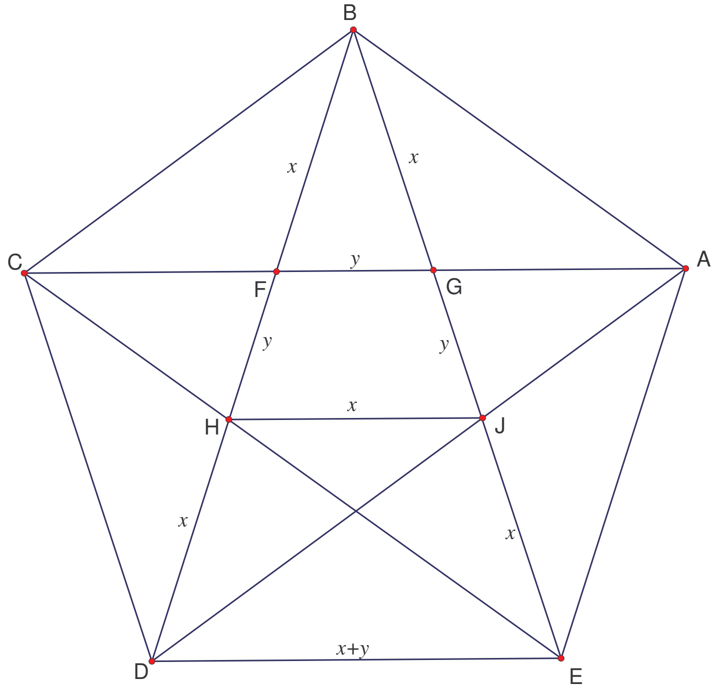


Note that the intersection of the diagonals form another regular pentagon inside the original one.

Now we know that the interior angles of a regular $n$-gon measure $\frac{n-2}{n}\pi$ radians, so that means that angle $\gamma$ has measure $\frac{3\pi}{5}$. Since $\beta$ is its supplement, it must have measure $\frac{2\pi}{5}$. Looking at the triangles, we have that $2\beta + \alpha = \pi$ and $2\theta + \gamma = \pi$, which tells us that $\alpha = \theta = \frac{\pi}{5}$. This means that $\Delta BCH \cong \Delta BHJ \cong \Delta BJA$.

(It should be noted that we could have found the above relationships by looking at the sides and diagonals rather than the angles: Draw $\overline{FJ}$, note that *FJHC* is a rhombus, and check out all the similarity that results from that.)

So now let's refocus our attention on three similar triangles in our sketch: $\Delta BFG$, $\Delta BHJ$, and $\Delta BDE$, and the length of their sides. If we set $x = BF$ and $y = FH$, we can express the lengths of all the segments in terms of $x$ and $y$, as in the following sketch:

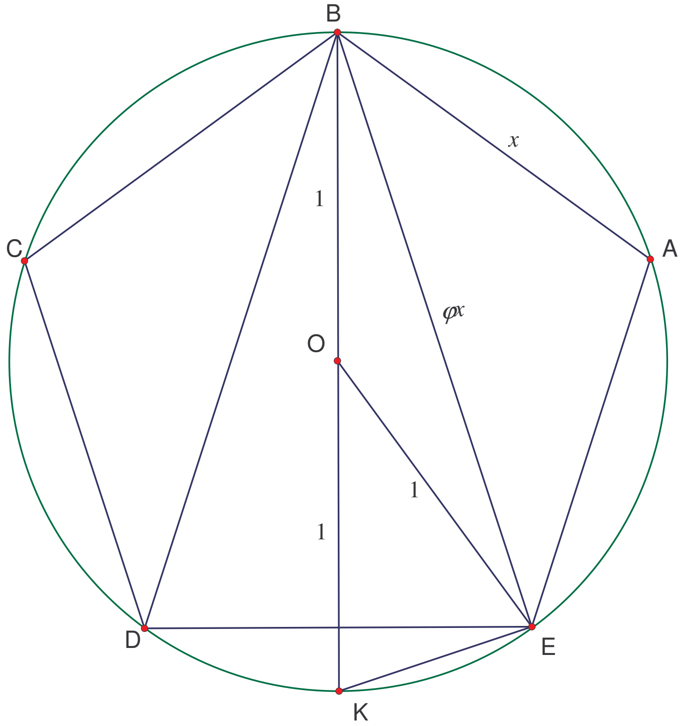


Because $\Delta BFG$ and $\Delta BHJ$ are similar, we get the following equation by taking the ratio of one of the legs over the base: $\frac{x}{y} = \frac{x+y}{x} \Rightarrow \frac{x}{y} = 1 + \frac{1}{x/y}$.

We can see that the ratio $\frac{x}{y}$ is none other than $\varphi$, the golden mean. So $\Delta BFG$ and $\Delta BHJ$ could be considered "golden triangles," since their sides are constructed from that ratio. Since $\Delta BDE$ is similar to $\Delta BFG$ and $\Delta BHJ$, it must also have the same ratio of sides. We see that indeed it does:

$$\frac{2x+y}{x+y} = 1 + \frac{x}{x+y} = 1 + \frac{y}{x} = 1 + \frac{1}{x/y}$$

So how does this help us inscribe a pentagon? Well, the first thing to notice is that the diagonals of a pentagon are $\varphi$ times as long as the sides. So if in the sketch below, we have a unit circle with a regular pentagon inscribed, and the sides of the pentagon are all of length $x$, then the diagonal $\overleftrightarrow{BE}$ has length of $\varphi x$.



Let's also notice that the measure of $\angle KBE$ must be $\frac{\pi}{10}$, since $\overline{BK}$ bisects $\angle DBE$. Then we can see that the central angle $\angle KOE$ must have measure $\frac{\pi}{5}$, since it is twice the measure of the inscribed angle $\angle KBE$. That means that $\Delta OKE$ is also a "golden triangle," since it's an isosceles triangle with a vertex angle of measure $\frac{\pi}{5}$.

And from our knowledge of golden triangles, this means that $KE = \frac{1}{\varphi}$.

Notice that because $\overleftrightarrow{BK}$ is a diameter of the circle, that $\Delta BEK$ is a right triangle, so

$$BK^2 = BE^2 + KE^2 \Rightarrow 2^2 = \varphi^2 x^2 + \frac{1}{\varphi^2} \Rightarrow x^2 = \frac{4}{\varphi^2} - \frac{1}{\varphi^4}.$$

Although this looks daunting at first, it's not that bad.

Recall that $\frac{1}{\varphi} = \varphi - 1$, so $\frac{1}{\varphi^2} = (\varphi - 1)^2 = \varphi^2 - 2\varphi + 1 = (\varphi + 1) - 2\varphi + 1 = 2 - \varphi$.

Therefore $\frac{1}{\varphi^4} = \left(\frac{1}{\varphi^2}\right)^2 = (2-\varphi)^2 = 4 - 4\varphi + \varphi^2 = 4 - 4\varphi + (\varphi + 1) = 5 - 3\varphi$.

So $x^2 = \frac{4}{\varphi^2} - \frac{1}{\varphi^4}$ becomes $x^2 = 4(2-\varphi) - (5 - 3\varphi) = 3 - \varphi$.

Now note that $x^2 = 3 - \varphi = (2-\varphi) + 1 = \frac{1}{\varphi^2} + 1^2$, which means that $x$ is the length of the hypotenuse of a right triangle with legs of length 1 and $\frac{1}{\varphi}$.

What this means is that in order to inscribe a pentagon in a circle, we have to come up with a right triangle where one of the legs has length 1 and the other one $\frac{1}{\varphi} = \varphi - 1$. The hypotenuse of such a triangle will have the same length as each side of a regular pentagon inscribed in a circle of radius 1.